

A note on α -IDT processes*

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Abstract

In this note, we introduce the notion of α -IDT processes which is obtained from a slight and fundamental modification of the IDT property. Several examples of α -IDT processes are given and Gaussian processes which are α -IDT are characterized. A kind example of this Gaussian α -IDT is the standard fractional Brownian motion. Also, we invest some links between the α -IDT property, with selfdecomposability, temporal selfdecomposability, stability and self similarity.

Keywords : IDT processes, Additive processes, Fractional Brownian motion, Gaussian processes, Selfsimilarity, Stability, Selfdecomposability.

1 Introduction

This introductory section is devoted to give the definition of α -IDT processes and to exhibit numerous examples of such processes. This notion of α -IDT is obtained from a slight and fundametal modification of the IDT property introduced by R. Mansuy [7]. This class of processes turn out to be also rich than the class of IDT processes since we shall prove that fractional brownian motion with any Hurst index $H \in (0; 1)$, are α -IDT. This is not true in general for IDT processes.

Definition 1.1 *Let $\alpha > 0$ and $X = (X_t; t \geq 0)$ a stochastic process. X is said to be an α -IDT process if, and only if, for any $n \in \mathbb{N}^*$ we have*

$$(X_{n^{1/\alpha}t}; t \geq 0) \stackrel{(law)}{=} (X_t^{(1)} + \dots + X_t^{(n)}; t \geq 0) \quad (1)$$

where $(X^{(i)})_{1 \leq i \leq n}$ are independent copies of X .

Remark 1.1 *For $\alpha = 1$, this definition is reduced to the well-known class of IDT processes (see R. Mansuy [7], K. Es-Sebaïy & Y. Ouknine [5], and A. Hakassou & Y. Ouknine [6]).*

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Let us now, provide many examples as possible of α -IDT processes which are not necessarily IDT processes.

Example 1.1 Let $0 < \alpha \leq 2$ and consider a strictly stable random variable S_α with parameter α . Then, the stochastic process X defined by:

$$X_t = tS_\alpha, t \in \mathbb{R}_+,$$

is an α -IDT process.

Example 1.2 Let $\alpha > 0$ and consider a strictly 1-stable random variable S . Then, the stochastic process X defined by

$$X_t = t^\alpha S, t \in \mathbb{R}_+,$$

is an α -IDT process.

Example 1.3 If X is an α -IDT process and μ a measure on \mathbb{R}_+ such that

$$X_t^{(\mu)} = \int \mu(du) X_{ut}, t \in \mathbb{R}_+$$

is well defined, then $X^{(\mu)}$ is again an α -IDT process.

2 Gaussian α -IDT processes

The next proposition emphasizes again how much the class α -IDT processes is rich, since we shall prove that fractional Brownian motion of any order $H \in (0, 1)$ are α -IDT with $\alpha = 2H$.

Proposition 2.1 Let $\alpha > 0$ and $(G_t; t \geq 0)$ be a centered Gaussian process which is assumed to be continuous in probability. Then the following properties are equivalent:

1. $(G_t; t \geq 0)$ is an α -IDT process.
2. The covariance function $c(s, t) := \mathbb{E}[G_s G_t]$, $0 \leq s \leq t$, satisfies

$$\forall a > 0, c(as, at) = a^\alpha c(s, t), \text{ for all } 0 \leq s \leq t.$$

3. The process $(G_t; t \geq 0)$ satisfies the scaling property of order $h = \frac{\alpha}{2}$, namely

$$\forall a > 0, (G_{at}; t \geq 0) \stackrel{(law)}{=} (a^{\alpha/2} G_t; t \geq 0).$$

4. The process $(\tilde{G}_y := e^{-\frac{\alpha}{2}y} G_{e^y}; y \in \mathbb{R})$ is stationary.
5. The covariance function $\tilde{c}(y, z) := \mathbb{E}[\tilde{G}_y \tilde{G}_z]$, $y, z \in \mathbb{R}$, is of the form

$$\tilde{c}(y, z) = \int \mu(du) e^{iu|y-z|}, y, z \in \mathbb{R}$$

where μ is a positive, finite, symmetric measure on \mathbb{R} .

Then, under these equivalent conditions, the covariance function c of $(G_t; t \geq 0)$ is given by

$$c(s, t) = (st)^{\alpha/2} \int \mu(da) e^{ia|\ln(\frac{s}{t})|}.$$

Proof.

(1 \Leftrightarrow 2) The α -IDT property (1) is equivalent to

$$\forall n \in \mathbb{N}, \quad c(n^{1/\alpha}s, n^{1/\alpha}t) = nc(s, t), \quad 0 \leq s \leq t$$

and also

$$\forall q \in \mathbb{Q}_+, \quad c(q^{1/\alpha}s, q^{1/\alpha}t) = qc(s, t), \quad 0 \leq s \leq t$$

The result is then obtained using the density of \mathbb{Q}_+ in \mathbb{R}_+ and the continuity of c (deduced by the continuity in probability, hence in L^2 , of $(G_t, t \geq 0)$).

(2 \Leftrightarrow 3) Since the law of a centered Gaussian process is determined by its covariance function, the result follows immediately.

(3 \Leftrightarrow 4) Apply Lamperti's transformation of order $h = \alpha/2$.

(4 \Leftrightarrow 5) Bochner's theorem for definite positive functions.

□

Example 2.1 Consider $B_H = (B_H(t); t \geq 0)$ the normalized fractional Brownian motion with Hurst index $H \in (0, 1)$, that is a centered continuous Gaussian process such that:

$$\mathbb{E}[B_H(t)B_H(s)] = \frac{1}{2}(|t|^{2H} + |s|^{2H} - |t - s|^{2H}), \quad 0 \leq s \leq t.$$

Then $B_H = (B_H(t); t \geq 0)$ is an $2H$ -IDT Gaussian process.

Example 2.2 Let $\varphi \in L^2(\mathbb{R}_+; du)$; Then the process G^φ defined by

$$\forall t > 0, G_t^\varphi = \int_0^\infty \varphi\left(\frac{u}{t}\right) dB_H(u), \quad \text{and } G_0^\varphi = 0,$$

where B_H is the normalized fractional Brownian motion with Hurst index $H \in (0, 1)$, is an $2H$ -IDT Gaussian process.

3 α -IDT and additive processes with the same marginals

Let us first note that the laws of finite-dimensional marginals of an α -IDT process are infinitely divisible. In particular, for any fixed t , the law of X_t is infinitely divisible.

Proposition 3.1 *Let $(X_t; t \geq 0)$ be a stochastically continuous α -IDT process. Denote by $(L_t; t \geq 0)$ the unique Lévy process such that*

$$X_1 \stackrel{law}{=} L_1.$$

Then $(X_t; t \geq 0)$ and the additive process $(L_{t^\alpha}; t \geq 0)$ have the same one-dimensional marginals, i.e. for any fixed $t \geq 0$,

$$X_t \stackrel{law}{=} L_{t^\alpha} \tag{2}$$

Proof.

From the definition of α -IDT process, it easy to check that for any $n \in \mathbb{N}^*$,

$$\mathbb{E} e^{i\theta X_{n^{1/\alpha}}} = (\mathbb{E} e^{i\theta X_1})^n \text{ for all } \theta \in \mathbb{R}$$

and also for any rational $q \in \mathbb{Q}_+$ that

$$\mathbb{E} e^{i\theta X_{q^{1/\alpha}}} = (\mathbb{E} e^{i\theta X_1})^q \text{ for all } \theta \in \mathbb{R}.$$

Now, thanks to the stochastic continuity of X and the density of \mathbb{Q}_+ in \mathbb{R}_+ , it follows that for any $t \geq 0$,

$$\mathbb{E} e^{i\theta X_{t^{1/\alpha}}} = (\mathbb{E} e^{i\theta X_1})^t \text{ for all } \theta \in \mathbb{R}.$$

Since $X_1 \stackrel{law}{=} L_1$ and thanks to the stochastic continuity of X and L , we get for any $t \geq 0$,

$$\mathbb{E} e^{i\theta X_{t^{1/\alpha}}} = \mathbb{E} e^{i\theta L_t} \text{ for all } \theta \in \mathbb{R}.$$

The proof is completed. □

Corollary 3.1 *If $(X_t; t \geq 0)$ is an α -IDT process, stochastically continuous with independent increments, then it is a time-changed Lévy process with the deterministic chronometer $h(t) = t^\alpha$.*

Proof.

The proof is consequence of Proposition 3.1 and Theorem 9.7 in K. Sato [8]. □

4 A link with path-valued Lévy processes

From the definition of an α -IDT process, it follows that viewed as a random variable taking values in path-space $D = D([0, \infty))$, i.e the classical Skorohod space of right-continuous paths over $[0, \infty)$, this random variable is infinitely divisible. We note that there exist D -valued infinitely divisible variables which are α -IDT. The main task of this section is to characterize D -valued infinitely divisible variable which are α -IDT.

Proposition 4.1 *Let X be a D -valued infinitely divisible variable. X is an α -IDT process if, and only if,*

- *its Lévy measure M over D satisfies for any non-negative functional F on D ,*

$$\int_D M(dy) F(y(n\cdot)) = n^\alpha \int_D M(dy) F(y(\cdot)) \quad (3)$$

- *its Gaussian measure ρ over D satisfies for any $v, u \in \mathbb{R}$,*

$$\int_D y(nu)y(nv)\rho(dy) = n^\alpha \int_D y(u)y(v)\rho(dy) \quad (4)$$

Proof.

The α -IDT property (1) admits the following equivalent formulation: for any $f \in C_c(\mathbb{R}_+, \mathbb{R}_+)$,

$$\mathbb{E}[\exp - \int_0^\infty dt f(t) X_{n^{1/\alpha}t}] = (\mathbb{E}[\exp - \int_0^\infty dt f(t) X_t])^n.$$

That is

$$\int_D M(dy) (1 - e^{-\int_0^\infty dt f(t) y(nt)}) = n^\alpha \int_D M(dy) (1 - e^{-\int_0^\infty dt f(t) y(t)})$$

and

$$\int_D \int_0^\infty du f(u) y(nu) \int_0^\infty dt f(t) y(nt) \rho(dy) = n^\alpha \int_D \int_0^\infty du f(u) y(u) \int_0^\infty dt f(t) y(t) \rho(dy)$$

from which (3) and (4) follow easily. □

Let us now provide some constructions of α -IDT processes.

Proposition 4.2 *Let N be a Lévy measure on path-space D . Define M as follows*

$$\int_D M(dy) F(y(\cdot)) = \int_0^\infty du \int_D N(dy) F(y(\frac{\cdot}{u^{1/\alpha}})) \quad (5)$$

Then M is an α -IDT Lévy measure, i.e. the Lévy measure of an α -IDT process viewed as a D -valued infinitely divisible random variable.

Proof.

Thanks to the trivial change of variable $u = n^\alpha v$, we get

$$\begin{aligned} \int_D M(dy) F(y(n\cdot)) &= \int_0^\infty du \int_D N(dy) F(y(\frac{n\cdot}{u^{1/\alpha}})) \\ &= n^\alpha \int_0^\infty dv \int_D N(dy) F(y(\frac{\cdot}{v^{1/\alpha}})) \\ &= n^\alpha \int_D M(dy) F(y(\cdot)) \end{aligned}$$

And the result is now obvious by Proposition 4.1. □

Example 4.1 Let X be a subordinator without drift, ν its Lévy measure, φ a regular function and define

$$X_t^{(\varphi)} = \int_0^\infty du \varphi(u) X_{(ut)^\alpha}$$

Then $(X_t^{(\varphi)}; t \geq 0)$ is an α -IDT process and its α -IDT Lévy measure satisfies for any functional over D ,

$$\int_D M^{(\varphi)}(dy) F(y(\cdot)) = \int_0^\infty du \int_D \nu(dx) F(x \Phi(\frac{u^{1/\alpha}}{\cdot}))$$

where Φ is the tail of the integral of φ : $\Phi(u) = \int_u^\infty dv \varphi(v)$.

5 Link with selfsimilarity and stability

Proposition 5.1 Let $(X_t; t \geq 0)$ be a non-trivial stochastically continuous α -IDT process. Then $(X_t; t \geq 0)$ is strictly β -stable if and only if it is (α/β) -selfsimilar.

Proof.

First, assume that $(X_t; t \geq 0)$ is (α/β) -selfsimilar. Then for all $n \in \mathbb{N}^*$ we have

$$(X_{n^{1/\alpha}t}; t \geq 0) \stackrel{(law)}{=} (n^{1/\beta} X_t; t \geq 0)$$

and using the α -IDT property, we obtain

$$(n^{1/\beta} X_t; t \geq 0) \stackrel{(law)}{=} (X_{n^{1/\alpha}t}; t \geq 0) \stackrel{(law)}{=} \left(\sum_{j=1}^{j=n} X_t^{(j)}; t \geq 0 \right)$$

where $X^{(1)}, \dots, X^{(n)}$ are independent copies of X . Thus X is strictly β -stable.

Conversely, suppose that $(X_t; t \geq 0)$ is strictly β -stable. Since $(X_t; t \geq 0)$ is α -IDT, it follows that for any $n \in \mathbb{N}^*$

$$(X_{n^{1/\alpha}t}; t \geq 0) \stackrel{(law)}{=} ((n^{1/\alpha})^{\alpha/\beta} X_t; t \geq 0)$$

and also for any rational $q \in \mathbb{Q}_+$,

$$(X_{q^{1/\alpha}t}; t \geq 0) \stackrel{(law)}{=} ((q^{1/\alpha})^{\alpha/\beta} X_t; t \geq 0).$$

Thanks the stochastic continuity, it follows that X is (α/β) -selfsimilar. □

Proposition 5.2 A non-trivial strictly β -stable and (α/β) -selfsimilar process $(X_t; t \geq 0)$ is an α -IDT process.

Proof.

Since X is strictly β -stable, we have for all $n \in \mathbb{N}^*$

$$(n^{1/\beta} X_t; t \geq 0) \stackrel{(law)}{=} \left(\sum_{j=1}^{j=n} X_t^{(j)}; t \geq 0 \right)$$

where $X^{(1)}, \dots, X^{(n)}$ are independent copies of X . On the other hand, it follows from the (α/β) -selfsimilarity of X that

$$(X_{n^{1/\alpha} t}; t \geq 0) \stackrel{(law)}{=} (n^{1/\beta} X_t; t \geq 0)$$

which implies that $(X_t; t \geq 0)$ is an α -IDT process. □

6 Subordination through α -IDT process

Proposition 6.1 *Let $(X_t; t \geq 0)$ be a Lévy process and $(\xi_t; t \geq 0)$ an α -IDT chronometer such that X and ξ are independent. Then, $(Z_t := X_{\xi_t}; t \geq 0)$ is an α -IDT process.*

Proof.

Let $\xi^{(l)}$, $l = 1, \dots, n$, be independent copies of ξ . Since X is independent of ξ , then for every $m \geq 1$ and $\theta = (\theta_1, \dots, \theta_m) \in \mathbb{R}^m$, we have

$$\begin{aligned} J(n, \theta) &:= \mathbb{E} \exp \left\{ i \sum_{k=1}^m < \theta_k, X(\xi_{n^{1/\alpha} t_k}) > \right\} \\ &= \mathbb{E} \left[\left(\mathbb{E} \exp \left\{ i \sum_{k=1}^m < \theta_k, X(s_k) > \right\} \right)_{s_k = \xi_{n^{1/\alpha} t_k}, k=1, \dots, m} \right]. \end{aligned}$$

Using the α -IDT property, we obtain

$$J(n, \theta) = \mathbb{E} \left[\left(\mathbb{E} \exp \left\{ i \sum_{k=1}^m < \theta_k, X(s_k) > \right\} \right)_{s_k = \sum_{l=1}^n \xi_{t_k}^{(l)}, k=1, \dots, m} \right].$$

According to the change of variables $\lambda_k = \theta_k + \dots + \theta_m$ and $t_0 = 0$, we have

$$J(n, \theta) = \mathbb{E} \left[\left(\mathbb{E} \exp \left\{ i \sum_{k=1}^m < \lambda_k, X(s_k) - X(s_{k-1}) > \right\} \right)_{s_k = \sum_{l=1}^n \xi_{t_k}^{(l)}, k=1, \dots, m} \right].$$

By the independence of increments of X , we get

$$J(n, \theta) = \mathbb{E} \left[\left(\prod_{k=1}^m \mathbb{E} \exp \left\{ i < \lambda_k, X(s_k) - X(s_{k-1}) > \right\} \right)_{s_k = \sum_{l=1}^n \xi_{t_k}^{(l)}, k=1, \dots, m} \right].$$

Now, it follows from the stationary of the increments of X and the independence of the $\xi^{(l)}$, $l = 1, \dots, n$, that

$$J(n, \theta) = \mathbb{E} \left[\left(\prod_{k=1}^m \prod_{l=1}^n \mathbb{E} \exp \left\{ i < \lambda_k, X(r_{l,k}) > \right\} \right)_{r_{l,k} = \xi_{t_k}^{(l)} - \xi_{t_{k-1}}^{(l)}, k=1, \dots, m; l=1, \dots, n} \right].$$

And that is

$$J(n, \theta) = \mathbb{E}[(\prod_{l=1}^n \mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, X(r_{l,k}) \rangle\})_{r_{l,k}=\xi_{t_k}^{(l)}, k=1, \dots, m; l=1, \dots, n}].$$

Now, this implies

$$J(n, \theta) = (\mathbb{E}[(\mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, X(\xi_{r_k}) \rangle\})_{r_k=\xi_{t_k}, k=1, \dots, m}])^n.$$

Hence

$$J(n, \theta) = (\mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, X(\xi_{t_k}) \rangle\})^n.$$

The proof is achieved. □

7 Links with temporal selfdecomposability and selfdecomposability

Proposition 7.1 *A stochastically continuous α -IDT process is temporally selfdecomposable of infinite order.*

Proof.

For any $(\theta_1, \dots, \theta_m) \in \mathbb{R}^m$, it follows thanks to the α -IDT property that for any $n \in \mathbb{N}^*$,

$$\mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, X_{t_k} \rangle\} = (\mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, X_{n^{1/\alpha} t_k} \rangle\})^{1/n}$$

And for any $q \in \mathbb{Q}_+^*$,

$$\mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, X_{t_k} \rangle\} = (\mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, X_{q^{1/\alpha} t_k} \rangle\})^{1/q}$$

In particular for $b \in \mathbb{Q}_+^* \cup (0; 1)$, we get

$$\begin{aligned} \mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, X_{t_k} \rangle\} &= (\mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, X_{b^{1/\alpha} t_k} \rangle\}) \\ &\quad \times (\mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, X_{b^{1/\alpha} t_k} \rangle\})^{\frac{1}{b}-1} \end{aligned}$$

That is

$$\begin{aligned} \mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, X_{t_k} \rangle\} &= (\mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, X_{b^{1/\alpha} t_k} \rangle\}) \\ &\quad \times (\mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, X_{(\frac{b}{b'})^{1/\alpha} t_k} \rangle\}) \text{ with } \frac{1}{b'} = \frac{1}{b} - 1. \end{aligned}$$

By stochastic continuity, it follows that for any $b \in (0; 1)$, we get

$$\begin{aligned} \mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, X_{t_k} \rangle\} &= (\mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, X_{b^{1/\alpha} t_k} \rangle\}) \\ &\quad \times (\mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, X_{(b/b')^{1/\alpha} t_k} \rangle\}) \end{aligned}$$

Now we set $c = b^{1/\alpha}$ and $c' = (b')^{1/\alpha}$, and then we deduce that $(X_t; t \geq 0)$ is temporally selfdecomposable and the c -residual has finite dimensional marginals which fit with those of the process X suitably rescaled, the result follows. \square

Proposition 7.2 *Let $(X_t; t \geq 0)$ be an α -IDT process. If X is selfdecomposable, then its residual is also an α -IDT process.*

Proof.

By definition (see O. E. Barndorff-Nielsen *et al* [2]), if X is selfdecomposable then for any $c \in (0; 1)$ there exists a process $U^{(c)} = (U_t^{(c)}; t \geq 0)$ called the c -residual of X , such that

$$X \stackrel{(law)}{=} cX' + U^{(c)}$$

where X' is an independent copie of X and X' and $U^{(c)}$ are independent. Thus it follows that for any $(\theta_1, \dots, \theta_m)$ we have

$$\mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, U_{t_k}^{(c)} \rangle\} = \frac{\mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, X_{t_k} \rangle\}}{\mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, cX'_{t_k} \rangle\}}$$

Now, for any $n \in \mathbb{N}^*$, we have thanks to the α -IDT property of X and X' :

$$\mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, U_{n^{1/\alpha} t_k}^{(c)} \rangle\} = \frac{(\mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, X_{t_k} \rangle\})^n}{(\mathbb{E} \exp\{i \sum_{k=1}^m \langle \theta_k, cX'_{t_k} \rangle\})^n}$$

And that is $U^{(c)}$ is an α -IDT process. \square

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